Dynamics of an Initially Stressed Fluid-Immersed Cylindrical Shell

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This investigation considers the axisymmetric dynamic response of an initially stressed elastic cylindrical shell submerged in a fluid. Differential equations of motion and dispersion relations for the fluid-shell system are presented. A transient solution is found for the motion of an unbounded shell excited by an impulsively applied radially directed ring loading.

Introduction

THE theory of the dynamic response of plates and shells submerged in a fluid has been treated extensively in the literature. A small and incomplete list of pertinent publications is provided by Refs. 1-10. When a closed shell is submerged in a fluid, it is subjected to normal fluid pressure acting on its surface. Static pressures of this kind will induce what we shall call initial stresses (or prestresses) and deformations. Subsequently applied static and time-dependent loads will result in incremental deformations and stresses in the shell. Initial stresses can be the cause of radical changes in the dynamical characteristics of the shell which result in significant differences between the transient response of shells with and without prestress. Since all investigations of shell-fluid interaction to date seem to neglect this important effect, it was felt that this phenomenon deserves to be investigated. In particular, thz present investigation considers the axisymmetric dynamic response of an initially stressed, elastic cylindrical shell submerged in a fluid. The shell is of unbounded length, with thickness h and mean radius a. It is subjected to static axial and radial prestress, and is subsequently subjected to radially directed transient loads.

Basic Equations

We shall refer the domain of the shell to cylindrical coordinates r, θ, z, such that the z axis is the axis of symmetry of the cylinder, and $a-h/2\leq r\leq a+h/2$, $0\leq\theta<2\pi$, $-\infty<z<+\infty$. The shell is immersed in a fluid which occupies the surrounding space $a+h/2<r$, $-\infty<z<+\infty$. The shell is assumed to be in a state of axially symmetric prestress characterized by the stress components $\tau^0_{rr}=\tau^0_{rr}(r)$, $\tau^0_{\theta\theta}=\tau^0_{\theta\theta}(r)$, and $\tau^0_{zz}=$ constant. It is assumed that incremental deformations referred to the initial, prestressed equilibrium configuration are characterized by

$$u_r=w(z,t)\qquad u_z=-\xi w_{,z}\qquad \xi=r-a \tag{1}$$

i.e., there is no "thickness stretch," Kirchhoff's hypothesis is satisfied, and the shell deforms in the inextensional manner.

The stress equations of incremental motion of the shell are now derived with the aid of Hamilton's principle, i.e.,

$$\int_{t_1}^{t_2}(\delta T-\delta U+\delta W)\,\mathrm{d}t=0 \tag{2}$$

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where T and U are the incremental kinetic and potential energy, respectively, and δW is the incremental virtual work. Using the results of Ref. 11, we have

$$\delta U=\int_V\sigma^{ij}(\delta u_i)\,|_j\mathrm{d}V=\int_0^{2\pi}\int_0^L\int_{-h/2}^{h/2}\Big[\sigma_{rz}(\delta w)_{,z}+r^{-1}\sigma_{\theta\theta}\delta w-\sigma_{zr}\delta w_{,z}-\sigma_{zz}\xi(\delta w_{,z})_{,z}\Big](a+\xi)\,\mathrm{d}\xi\cdot\mathrm{d}z\cdot\mathrm{d}\theta \tag{3}$$

where

$$\left.\begin{aligned}\sigma_{rz}&=\tau_{rz}+\tau^0_{rr}w_{,z} & \sigma_{zr}&=\tau_{zr}-\tau^0_{zz}w_{,z}\\ \sigma_{\theta\theta}&=\tau_{\theta\theta}-\tau^0_{\theta\theta}w_{,zz} & \sigma_{zz}&=\tau_{zz}+\tau^0_{zz}r^{-1}w\end{aligned}\right\} \tag{4}$$

Upon substitution of Eq. (4) into Eq. (3), integration by parts, and utilization of Eq. (6), we obtain

$$\begin{aligned}\delta U&=2\pi a[-M_{zz}\delta w_{,z}+(Q_z+N^0_{rr}w_{,z})\delta w]_0^L\\&-2\pi a\int_0^L\Big[-(M_{zz,z}-Q_z+N^0_{zz}w_{,z})\delta w_{,z}\\&+(Q_{z,z}+N^0_{rr}w_{,zz}-a^{-1}N_{\theta\theta}+a^{-1}M^0_{\theta\theta}w_{,zz})\delta w\Big]\mathrm{d}z\end{aligned} \tag{5}$$

where the stress resultants are defined by

$$M_{zz}=\int_{-h/2}^{h/2}\tau_{zz}\Big(1+\frac{\xi}{a}\Big)\xi\mathrm{d}\xi\qquad Q_z=\int_{-h/2}^{h/2}\tau_{rz}\Big(1+\frac{\xi}{a}\Big)\mathrm{d}\xi$$

$$N_{\theta\theta}=\int_{-h/2}^{h/2}\tau_{\theta\theta}\mathrm{d}\xi \tag{6a}$$

$$N^0_{rr}=\int_{-h/2}^{h/2}\tau^0_{rr}\Big(1+\frac{\xi}{a}\Big)\mathrm{d}\xi\qquad N^0_{zz}=\int_{-h/2}^{h/2}\tau^0_{zz}\mathrm{d}\xi=h\tau^0_{zz}$$

$$M^0_{\theta\theta}=\int_{-h/2}^{h/2}\tau^0_{\theta\theta}\xi\mathrm{d}\xi \tag{6b}$$

In view of Eq. (1), the time integral of the variation of the incremental kinetic energy is given by

$$\begin{aligned}\int_{t_1}^{t_2}\delta T\mathrm{d}t&=-\rho_s\int_{t_1}^{t_2}\int_V\ddot{u}^i\delta u_i\mathrm{d}V\mathrm{d}t\\&=-\rho_s\int_{t_1}^{t_2}\int_0^L\int_{-h/2}^{h/2}\int_0^{\pi}\ddot{u}_r\delta u_r(a+\xi)\,\mathrm{d}\theta\mathrm{d}\xi\mathrm{d}z\mathrm{d}t\\&=-2\pi\rho_s\int_{t_1}^{t_2}\int_0^L ah\ddot{w}\delta w\mathrm{d}z\mathrm{d}t\end{aligned} \tag{7}$$

where we have neglected the rotatory inertia term. The virtual work of external, incremental forces is $\delta W = \int_S T^i \delta u_i dS = \int_S T \cdot \delta u dS$, where $u = \hat{e}_r u_r + \hat{e}_z u_z$, $T = \hat{e}_r T_r + \hat{e}_z T_z$, and where S denotes the surfaces of the shell.

Consequently

$$\delta W = 2\pi\left(a + \frac{h}{2}\right)\int_0^L p\delta w dz + 2\pi a (Q_z^* \delta w + M_{zz}^* \delta w_{,z})\Big|_0^L \quad (8)$$

where $p \equiv T_r$ on the cylindrical surface $r = a + h/2$, $0 < z < L$, and

$$\int_{-h/2}^{h/2} T_r (a+\xi) d\xi = aQ_z^*$$

$$\int_{-h/2}^{h/2} T_z (a+\xi) d\xi = aN_{zz}^* \qquad \int_{-h/2}^{h/2} T_z (a+\xi)\xi d\xi = aM_{zz}^*$$

where Q_z^*, N_{zz}^*, and M_{zz}^* denote values of the shear force, axial force, and bending moment per unit length at $z=0$ and $z=L$. We now substitute Eqs. (5, 7, and 8) into Eq. (2), and observe that the length of the shell L, the time interval $t_2 - t_1$, as well as the variation δw are arbitrary. This results in the stress equations of incremental motion

$$Q_{z,z} + N_{rr}^0 w_{,zz} - a^{-1} N_{\theta\theta} - a^{-1} M_{\theta\theta}^0 w_{,zz} + (1 + h/2a) p = \rho_s h \ddot{w} \quad (9a)$$

$$Q_z = M_{zz,z} + N_{zz}^0 w_{,z} \quad (9b)$$

and the associated admissible boundary conditions which require the specification at $z=0$ and at $z=L$ of either w or $Q_z + N_{rr}^0 w_{,z}$, and either M_{zz} or $w_{,z}$. We now assume that the incremental deformation of the shell is characterized by a linear, isotropic constitutive relation. If we neglect the effect of τ_{rr}, then it can be shown that

$$M_{zz} = -Dw_{,zz} \qquad N_{\theta\theta} = Ehw/a(1-\nu^2) \quad (10)$$

where $D = Eh^3/12(1-\nu^2)$ and $(h/2a)^2 \ll 1$. Combining Eqs. (9) and (10), we readily obtain

$$Dw_{,zzzz} - Nw_{,zz} + bw + mw_{,tt} = q \quad (11)$$

where $N = N_{zz}^0 + N_{rr}^0 + a^{-1} M_{\theta\theta}^0$, $q = (1 + h/2a)p$, $b = Eh/a^2(1-\nu^2)$, and $m = \rho_s h$. We note that the term characterized by N incorporates all admissible prestress parameters. This includes the terms N_{rr}^0 and $M_{\theta\theta}^0$ which are caused by the hydrostatic pressure exerted by the surrounding fluid upon the shell.

In the case of irrotational axisymmetric flow referred to cylindrical coordinates, the fluid velocity components are derivable from a scalar potential $\phi = \phi(r, z, t)$ such that

$$v_r = -\phi_{,r} \qquad v_z = -\phi_{,z} \quad (12)$$

where ϕ satisfies the wave equation for a compressible (acoustic) fluid

$$\phi_{,zz} + \phi_{,rr} + r^{-1}\phi_{,r} = c^{-2}\phi_{,tt} \quad (13)$$

and where c is the speed of sound in the fluid. For an inviscid fluid, the incremental pressure is derivable from the velocity potential by the relation

$$p = -\rho_F \phi_{,t} \quad (14)$$

We shall assume that the transient disturbance acting on the shell consists of a radially directed line load applied impulsively at $z=0$ and at $t=0$. In addition, we must account for the reaction of the fluid to the motion of the shell. In view of Eq. (14), we have

$$q(z,t) = I_0 \delta(z)\delta(t) - \rho_F \phi_{,t}(a,z,t) \quad (15)$$

We also require that the fluid remains in contact with the shell, i.e., $w_{,t} = v_r(a,z,t)$ or, using Eq. (12),

$$w_{,t}(z,t) = -\phi_{,r}(a,z,t) \quad (16)$$

Combining Eqs. (15) and (11), we obtain the final form of the equation of incremental motion of the fluid immersed shell:

$$Dw_{,zzzz} - Nw_{,zz} + bw + mw_{,tt} = I_0\delta(z)\delta(t) - \rho_F\phi_{,t}(a,z,t) \quad (17)$$

Initially, at $t=0$, the shell as well as the surrounding fluid is in a state of static equilibrium. Consequently,

$$w(z,0) = w_{,t}(z,0) = 0 \quad (18)$$

$$\phi(r,z,0) = \phi_{,t}(r,z,0) = 0 \quad (19)$$

In addition, we require

$$w = w_{,z} = w_{,zz} = w_{,zzz} = 0 \text{ as } z \to \pm\infty \quad (20)$$

$$\phi = \phi_{,z} = 0 \text{ as } z \to \pm\infty \quad (21)$$

$$\lim_{r\to\infty} \phi(r,z,t) = \alpha < \infty \quad (22)$$

Dispersion Relations

We now proceed to find the dispersion relations for the shell-fluid system. Assuming traveling, time-harmonic waves with frequency ω and wave number k, we have

$$\phi = BK_0(\eta r) \exp i(kz - \omega t) \quad (23a)$$

$$w = A \exp i(kz - \omega t) \quad (23b)$$

where $\eta = (k^2 - \omega^2/c^2)^{1/2}$ and K_n denotes the Bessel function of the second kind, of order n. We note that $\phi \to 0$ as $\gamma \to \infty$, as required in the present case. For this reason we need not consider the corresponding Bessel function of the first kind, since it does not vanish for unbounded arguments. Equations (23) are now substituted into Eqs. (13) and (17) (with $I_0 \equiv 0$), and we use Eq. (16). The result can be expressed by

$$K^4 + 2\beta K^2 + 1 - \Omega^2[1 + \Delta K_0(\theta H)/HK_1(\theta H)] = 0 \quad (24)$$

or, equivalently, by

$$K^2 + 2\beta + K^{-2} - V^2[1 + \Delta K_0(\theta H)/HK_1(\theta H)] = 0 \quad (25)$$

where

$$X = (b/D)^{1/4} z, \qquad K = (D/b)^{1/4} k, \qquad \Omega = (m/b)^{1/2}\omega$$

$$\theta = (12)^{1/4}(a/h)^{1/2}, \; H = (K^2 - \Omega^2/C^2)^{1/2} = K(1 - V^2/C^2)^{1/2}$$

$$\beta = -N/N_{cr}, \qquad N_{cr} = -2(bD)^{1/2}$$

$$V = \Omega/K = (12)^{1/4}(\rho_s/E)^{1/2}(a/h)^{1/2} v$$

$$C = (12)^{1/4}(\rho_s/E)^{1/2}(a/h)^{1/2}(1-\nu^2)^{1/2} c$$

$$\Delta = (12)^{-1/4}(\rho_F/\rho_s)(a/h)^{1/2} \quad (26)$$

The dispersion relation Eq. (25) is an implicit relationship between phase velocity V and wave number K. Equation (25) yields only a single positive real root V for each choice of $K \geq 0$. This fact is consistent with the special constraints of our shell model, i.e., there is only a single mode of (flexural) wave propagation possible, this being a direct consequence of our basic assumptions, Eq. (1). We note that C characterizes the compressibility of the fluid. If we let $C \to \infty$ in Eq. (25), we obtain the dispersion relation for an incompressible fluid:

$$V = G/KL \qquad \left.\begin{array}{l} G^2 = K^4 + 2\beta K^2 + 1 \\ L^2 = 1 + \Delta K_0(\theta K)/KK_1(\theta K) \end{array}\right\} \tag{27}$$

The following values are assumed (steel shell surrounded by water):

$$\left.\begin{array}{lll} \rho_F/\rho_s = 0.127 & \nu = 0.3 & a/h = 50 \\ (E/\rho_s)^{1/2} = 16{,}900 \text{ fps} & & c = 4870 \text{ fps} \end{array}\right\} \tag{28}$$

Figure 1 shows a nondimensional graph of the dispersion relation V vs K for a compressible fluid [Eq. (25)] and an equivalent incompressible fluid [Eq. (27)]. We note that the differences are small for the wave number range $0.2 < K < 3.4$. These differences are weakly dependent upon the value of the prestress parameter β, but studies have shown that they decrease as $\beta \to -1$.

Transient Solution

We now proceed to the solution of the problem characterized by Eqs. (17-22) in conjunction with Eqs. (13) and (16). We take Fourier transforms (star) with respect to z, and Laplace transforms (bars) with respect to t. In this manner Eqs. (11, 13, and 16) are transformed into

$$Dk^4 \bar{w}^* + Nk^2 \bar{w}^* + b\bar{w}^* + ms^2 \bar{w}^*$$

$$= I_0(1)(2\pi)^{-1/2} - \rho_F s \bar{\phi}^*(a) \tag{29}$$

$$\bar{\phi}^*_{,rr} + r^{-1}\bar{\phi}^*_{,r} - (k^2 + s^2/c^2)\bar{\phi}^* = 0 \tag{30}$$

$$s\bar{w}^* = -\bar{\phi}^*_{,r}(a,k,s) \tag{31}$$

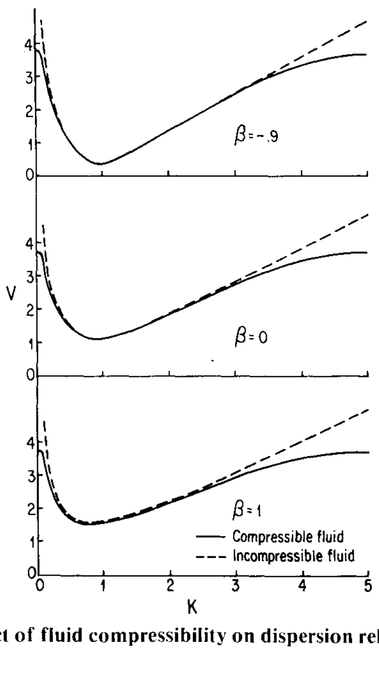

Fig. 1 Effect of fluid compressibility on dispersion relations of shell-fluid system.

where $\bar{w}^* = \bar{w}^*(k,s)$, $\bar{\phi}^* = \bar{\phi}^*(r,k,s)$, and where k and s are the Fourier and Laplace transform parameters, respectively. The solution of Eq. (30) which remains bounded for $r \to \infty$ is given by $\bar{\phi}^* = AK_0(\mu r)$. Upon substitution into Eq. (31), we obtain

$$s\bar{w}^* = -\frac{d}{dr}\Big[AK_0(\mu r)\Big]_{r=a} = A\mu K_1(\mu a)$$

so that $A = (s/\mu)\bar{w}^*/K_1(\mu a)$. Consequently, the velocity potential in transform space becomes

$$\bar{\phi}^* = (s/\mu)\,[K_0(\mu r)/K_1(\mu a)]\,\bar{w}^* \tag{32}$$

Combining Eqs. (32) and (29), we obtain the transient solution for the shell deflection in transform space:

$$[(2\pi)^{1/2}/\Gamma]\,\bar{w}^* = (G^2 + S^2F^2)^{-1} \tag{33}$$

where

$$G^2 = K^4 + 2\beta K^2 + 1$$
$$F^2 = 1 + \Delta K_0(\theta M)/MK_1(\theta M)$$
$$\Gamma = (I_0/b)(b/D)^{1/4}(b/m)^{1/2}$$
$$M = (K^2 + S^2/C^2)^{1/2}$$
$$S = (m/b)^{1/2}s$$

Inversion of Eq. (33) is tedious. However, if at this point, we assume that compressibility effects in the fluid can be neglected, then Eq. (33) reduces to

$$(\sqrt{2\pi}/\Gamma)\,\bar{w}^* = (G^2 + S^2L^2)^{-1} \tag{34}$$

where L^2 is given by Eq. (27). Successive inversions of Eq. (34) with respect to S and K, respectively, result in

$$W(X,T) = \frac{1}{\pi}\int_0^\infty \frac{KV \sin KVT}{K^4 + 2\beta K^2 + 1}\cos KxdK \tag{35}$$

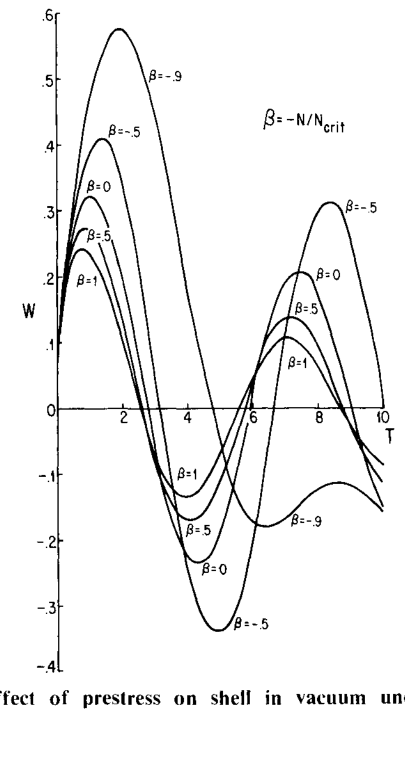

Fig. 2 Effect of prestress on shell in vacuum under impulsive loading.

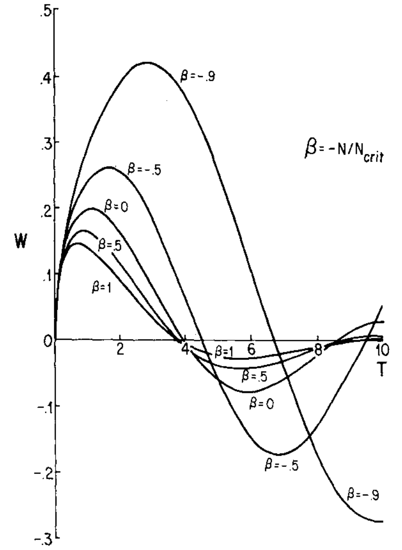

Fig. 3 Effect of prestress on immersed shell under impulsive loading.

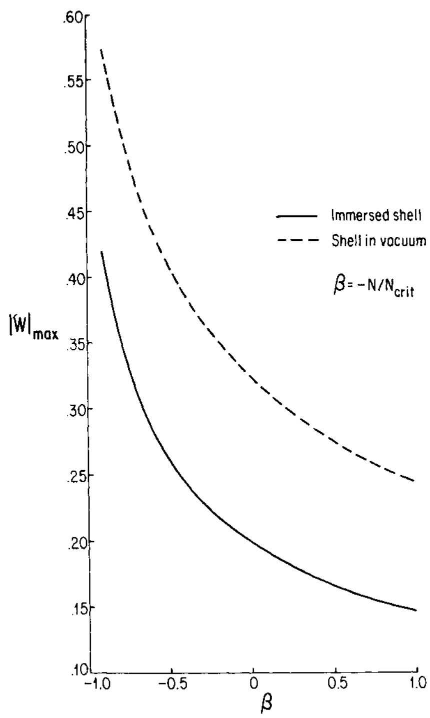

Fig. 4 Effect of surrounding fluid on maximum deflection of shell under impulsive loading.

where $V=V(K)$ is given by Eq. (27) and $T=(b/m)^{1/2}\ t$, $X=(b/D)^{1/4}$, and $W=w\Gamma^{-1}$. The integral in Eq. (35) is readily evaluated with the aid of a digital computer for any special value of X and T. Although the range of integration extends over the entire wave number spectrum $0\leq K<\infty$, only the range $0.2\leq K\leq 3.4$ was required for a sufficiently accurate evaluation of the integral, i.e., the integrand function of Eq. (35) is approximately equal to zero for $K\leq 0.2$ and $K\geq 3.4$. An inspection of Fig. 1 reveals that for the wave number spectrum $0.2\leq K\leq 3.4$, the dispersion relations for a compressible and incompressible fluid do not differ significantly. Moreover, the transient response characterized by Eq. (35) consists of a superposition of time harmonic, propagating waves emanating from the disturbance at $z=0$ (see Ref. 12). For these reasons it can be inferred that the incompressible fluid assumption and its consequence, Eq. (35), should yield a reasonably accurate approximation in the present case.

The curves in Figs. 2 through 4 were obtained for $X=0$ and the numerical values in Eq. (28). Figures 2 and 3 are graphs of $W(0,T)$ vs T, with prestress β as a parameter. Fig. 2 pertains to a steel shell in vacuum, while Fig. 3 describes the motion of the same shell in water. We note that compressive prestress increases the amplitude as well as the period of oscillation. Moreover, attenuation of the resulting motion with respect to time decreases with an increase in prestress. Fluid immersion increases the period of oscillation and decreases the magnitude of the deflection for all values of prestress. Figure 4 provides a comparison of maximum deflection at $X=0$ in water and in vacuum as a function of prestress. The effect of the fluid on the peak value of deflection increases with β, i.e., the maximum deflection of the shell in water is approximately 65% of the corresponding maximum deflection in vacuum for $\beta=1$, but only 35% greater in the case $\beta=-0.9$. Figure 4 clearly shows the nonlinear dependence of maximum deflection upon prestress, with deflections approaching unbounded values as the prestress approaches the critical (buckling) value $\beta=-1$. It also shows that the dynamic deflection of the shell is substantially reduced by water immersion for all possible values of prestress β.

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References

[1] Mindlin, R.P. and Bleich, H.H., "Response of an Elastic Cylindrical Shell to a Transverse, Step Shock Wave," *Journal of Applied Mechanics*, Vol. 20, June 1953, pp. 189-195.

[2] Haywood, J.H., "Response of an Elastic Cylindrical Shell to a Pressure Pulse," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. 11, 1958, pp. 129-141.

[3] Forrestal, M.J., "Response of an Elastic Cylindrical Shell to a Transverse, Acoustic Pulse," *Journal of Applied Mechanics*, Vol. 35, Sept. 1968, pp. 614-616.

[4] Geers, T.L., "Excitation of an Elastic Cylindrical Shell by a Transient Acoustic Wave," *Journal of Applied Mechanics*, Vol. 36, Sept. 1969, pp. 459-469.

[5] Payton, R.G., "Transient Interaction of an Acoustic Wave with a Circular Cylindrical Elastic Shell," *Acoustical Society of America Journal*, Vol. 32, 1960, pp. 722-729.

[6] Huang, H., "Exact Analysis of the Transient Interaction of Acoustic Plane Waves with a Cylindrical Elastic Shell," *Journal of Applied Mechanics*, Vol. 37, Dec. 1970, pp. 1091-1099.

[7] Berger, B.S., "Dynamic Response of an Infinite Cylindrical Shell in an Acoustic Medium," *Journal of Applied Mechanics*, Vol. 36, June 1969, pp. 342-345.

[8] Herrman, G. and Russell, J.E., "Forced Motion of Shells and Plates Surrounded by an Acoustic Fluid," *Proceedings of a Symposium on the Theory of Shells to Honor Lloyd Hamilton Donnell*, University of Houston, 1967, pp. 313-342.

[9] Junger, M.C. and Feit D., *Sound, Structures, and their Interaction*, M.I.T. Press, Cambridge, 1972.

[10] Reismann, H. and Jurney, W.H., "Effect of Water Immersion on Reduction of Die Stresses in Explosive Forming," *Journal of Engineering for Industry, Transactions of ASME, Ser. B*, Vol. 88, Feb. 1966, pp. 111-116.

[11] Reismann, H. and Pawlik, P., "Dynamics of Initially Stressed Hyperelastic Solids," *Solid Mechanics Archives*, Vol. 2, May 1977, pp. 129-185.

[12] Reismann, H. and Pawlik, P.S., "Appendix B: Kinematics of Wave Motion," *Elastokinetics*, West Publishing Co., St. Paul, 1974.